UNIVERSITÉ DE MONTRÉAL

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ON THE COMBINED DISTRIBUTION-ASSIGNMENT OF TRAFFIC

By

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This research was supported by a grant from the Ford Motor Company of Canada for applied research in the Analysis and Control of Urban Transportation Network.

Université de Montréal
Publication # 177 - Département d'informatique
Publication # 8 - Centre de recherche sur les transports
mai 1974
Révisé novembre 1974

Dépôt légal - 1er trimestre 1978

ABSTRACT

It has been remarked that in the practice of transportation planning the distribution and assignment computations are treated sequentially as independent problems, although ideally they should be solved simultaneously. A case has been made by many practitioners for repeating the distribution and assignment computations in order to obtain more consistent results. The purpose of this paper is to shed some light on the eventual results of sequentially solving the distribution and assignment problem. Our conclusion, based on rigorous arguments, is that such practice does not lead to valid results. Also, we develop an approach to solving the distribution and assignment problem simultaneously and relate it to other methods that were proposed for this problem. A numerical example illustrates the conclusions drawn.

RESUME

Dans la planification en transport, les phases de distribution et de l'affectation du trafic sont traditionnellement traitées de façon indépendante, bien que l'on reconnaisse qu'elles doivent s'intégrer dans une seule phase de calcul. En pratique on cycle plusieurs fois sur les phases de distribution et de l'affectation afin d'obtenir des résultats plus consistants. Le but de cet article est d'apporter des éclaircissements sur le traitement séquentiel des phases de distribution et de l'affectation du trafic. Notre conclusion est qu'une telle pratique ne produit pas nécessairement de meilleurs résultats. Nous présentons ensuite une méthode de résolution d'un modèle intégré de distribution et de l'affectation du trafic, et des discussions sur les méthodes existantes. Quelques résultats numériques sont également rapportés.

1. Introduction

It has been remarked [13], [14] that, in the practice of the transportation planning process, the distribution and assignment modules are treated sequentially as independent problems, although ideally they should be solved simultaneously. It is customary to use the accessibility time between origins and destinations to estimate the origin-destination displacements, and then assign these trip estimates on the model of the road network, taking into account the congestion effects. When the network is congested, as often is the case in important urban areas, it is quite likely that the accessibility times inferred from the assignment differ from those initially supposed for the distribution model. As a consequence, a case has been made by many practitioners for repeating the distribution and assignment computations, with the hope that the resulting estimates of traffic flows will be "more" consistent. In their recent and very good text, Potts and Oliver [13] recognize the problem and remark that "several attempts have been made to combine distribution with assignment... the simplest (of these attempts) resort to repeated use of distribution and assignment programs".

The purpose of this paper is to shed some light on the eventual results of repetitive use of distribution and assignment computations. Our conclusion, based on rigorous arguments, is that such practice does not always result in "more" consistent estimates, and, in general does

not converge to some global solution of distribution and assignment. This conclusion is perhaps not surprising to some experienced traffic planners, however we will give a theoretical justification of this fact.

A potential solution method for this problem is the work of Tomlin [14], who proposed a solution to the combined distribution and assignment model, based on the use of Dantzig-Wolfe [4] decomposition. Since both the distribution problem and the assignment problem have been formulated as mathematical programming problems, such an approach is attractive. We would like to suggest some straightforward, but useful modifications in the formulation of the assignment model used in [14] and propose, an alternate approach which also results in a decomposition of the problem. We develop in Section 4 of this paper an adaptation of the Frank and Wolfe [7] algorithm to this problem. Solution algorithms for the combined problem have been suggested by Irwin and Von Cube [10], Bruynooghe [1] and Evans [5]. We shall return to interpret their work in a later section of the paper.

2. The Combined Distribution-Assignment Problem

We introduce the following notation, which is similar to that used in [14]. Let O_p, denote the production of trips at origins $p = 1,\ldots,m$ and D_q, the attraction of trips at destinations $q = 1,\ldots,n$.

f_{pq} are the interzonal flows and c_{pq} are the travel access times from origin p to destination q. The road network is usually represented by a set of nodes N and a set of links A; the link delay (or supply) functions are $s_a(v_a)$, $a \epsilon A$, where v_a are the link flows. On the network, f_{pq} is divided into path flows $h_{k,pq}$. The combined problem is that of finding f_{pq} and $h_{k,pq}$ that solve

$$\text{Min} \sum_p \sum_q \gamma f_{pq} \ln f_{pq} + \sum_a \int_0^{v_a} s_a(v)\, dv \qquad (1)$$

subject to

$$\sum_q f_{pq} = O_p \quad , \quad \text{all } p \qquad (2)$$

$$\sum_p f_{pq} = D_q \quad , \quad \text{all } q \qquad (3)$$

$$-f_{pq} + \sum_k h_{k,pq} = 0 \quad , \quad \text{all } (p,q) \qquad (4)$$

$$v_a = \sum_p \sum_q \sum_k \delta_{ak,pq}\, h_{k,pq} \quad , \quad \text{all } a \qquad (5)$$

where

$$\delta_{ak,pq} = \begin{cases} 1 & \text{if } a \text{ belongs to path } k \text{ for pair } (p,q) \\ 0 & \text{otherwise} \end{cases}$$

$$h_{k,pq} \quad , \quad f_{pq} \geq 0$$

In the following we shall use $S_a(v_a) = \int_0^{v_a} s_a(v)dv$

We have, as in [14], used a formulation of the distribution problem as a pseudo-thermodynamic equilibrium problem, which Murchland [10] and more recently Evans [6] have

showed to be equivalent to a form of the gravity model. Unlike Tomlin, we have used a nonlinear (convex) programming formulation of the assignment problem such as studied by Dafermos [3] and others, where $s_a(v_a)$ are increasing convex functions that model the congestion volume-delay relationship. Such curves, sometimes referred to as capacity functions, have been used in practice [10], and are well accepted now. Next we turn our attention to interpret the repeated use of distribution and assignment calculations.

3. Solution by Partitioning and Interpretation

The notion associated with repetitive, sequential solution of the distribution and assignment problems is that of partitioning. A method for solving mathematical programs that are coupled by constraints such as (4), usually referred to as coupling constraints, was first given by Benders and then extended to nonlinear programs by Geoffrion [8]. The nontrivial adaptation of generalized Benders partitioning to our problem results in the following iterative algorithm:

Step 1 Select some reasonable c_{pq} for all (p,q) ; set $J = 0$, $UBD = \infty$, solve the distribution problem

$$\text{Min} \quad \sum_p \sum_q f_{pq} (c_{pq} + \gamma \ln f_{pq}) \tag{6}$$

$$\text{Subject to} \quad \sum_q f_{pq} = O_p \ , \ \sum_p f_{pq} = D_q \ , \ f_{pq} \geq 0 \tag{7}$$

and obtain interzonal flows $\bar{f}^{o}_{pq}$.

Step 2 Solve the assignment problem with $\bar{f}^J_{pq}$ as demands

$$\text{Min} \quad \sum_a S_a(v_a) \tag{8}$$

$$\text{Subject to} \quad \sum_k h_{k,pq} = \bar{f}^J_{pq} \quad , \quad h_{k,pq} \geq 0 \tag{9}$$

and v_a defined as before;

obtain path flows $\bar{h}^J_{k,pq}$ and $\bar{u}^J_{pq}$, the marginal costs of the utilized paths in the assignment.

Put $\text{UBD} = \min \{\text{UBD}, \sum_a S_a(\bar{v}^J_a)\}$, where $\bar{v}^J_a = \sum_a \delta_{ak,pq} \bar{h}^J_{k,pq}$.

Step 3 Solve the problem

$$\text{Min} \quad y \tag{10}$$

$$\text{Subject to} \quad y \geq \sum_p \sum_q f_{pq} (\gamma \ln f_{pq} + \bar{u}^j_{pq}) + M^j \quad , \quad j = 1,2,\ldots,J \tag{11}$$

$$\sum_q f_{pq} = O_p \quad , \quad \sum_p f_{pq} = D_q \quad , \quad f_{pq} \geq 0$$

where M^j is a constant.

If $y \geq \text{UBD} - \varepsilon$, terminate, otherwise set $J = J + 1$ and return to Step 2 ($\varepsilon \geq 0$ is a suitable convergence parameter).

The interpretation of the above iterative scheme is rather simple. A form of the distribution problem and the assignment problem are solved repetitively until convergence to the equilibrium values of f_{pq} and $h_{k,pq}$ is obtained. The problem in Step 2 is an ordinary

assignment problem however the problem in Step 3 is difficult to solve. That problem is a distribution problem when $j = 1$, (with $\bar{u}_{pq}$ having the role of the c_{pq}); however when $j \geq 2$, it is a difficult nonlinear programming problem with J convex constraints (11). Although we do not propose a solution method for the problem in Step 2, nor propose to use this computation scheme in practice, we are in a position to draw some powerful conclusions.

The ad hoc procedure of repetitive solution of the distribution and assignment models, is equivalent to applying the partitioning scheme outlined above, but dropping each time the constraints(11) from the previous iterations. Since the convergence of the computations depends on the presence of all these constraints, it is clear that the ad hoc procedure will not converge to the correct solution value, and should not be expected to converge at all. It is perhaps surprising to note that the ad hoc procedure is identical with solution by partitioning for the first iteration only, that is where $j = 1$. This may present a weak justification for carrying out the ad hoc procedure for an iteration, that is, having solved the assignment, resolve the distribution problem once, and redo the assignment. However, we would like to reemphasize that the justification for doing so is not too well founded.

We illustrate the nonconvergence of the sequential computations with a small numerical example that is given in Figure 1. There are 2

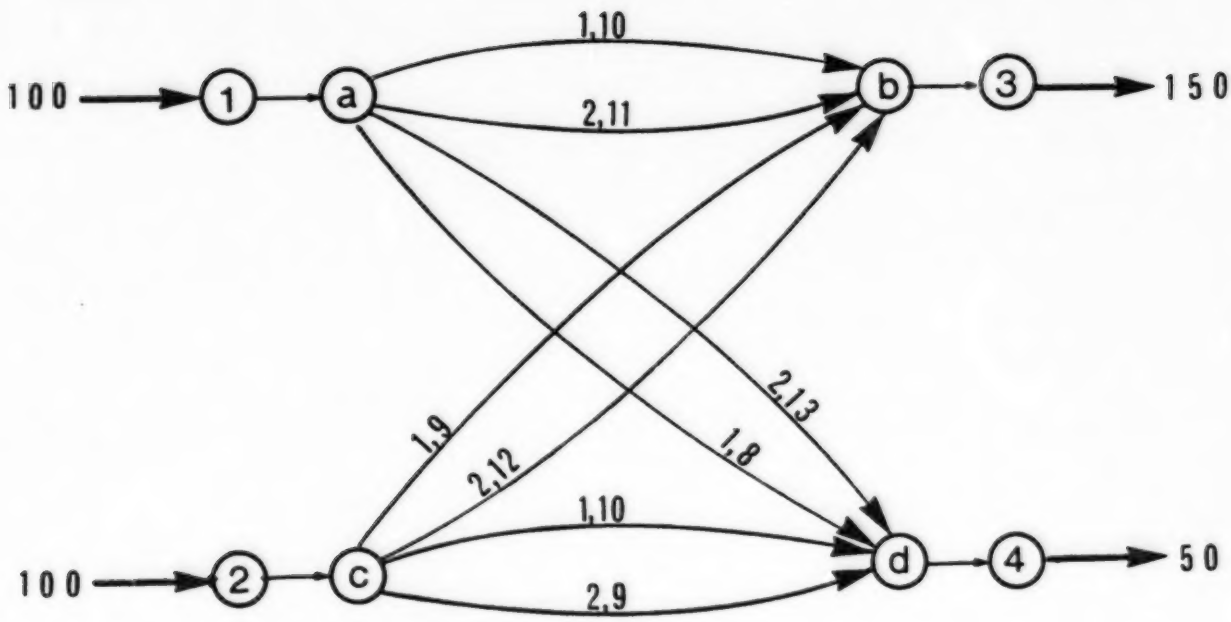

Figure 1

origins, nodes 1 and 2, that originate 100 trips each and 2 destinations, nodes 3 and 4, that have attractions of 150 and 50 respectively. The network is composed of eight links. The link volume-delay curves are assumed to be linear cost functions of the form

$$t_a = k_1 + k_2 v_a \quad , \quad \text{for all arcs } a \, .$$

The parameters $\{k_1 , k_2\}$ are indicated on each link. The parameter γ has been chosen to be 1. As it can easily be seen, the assignment costs have been chosen to be much higher than the distribution costs in order to accent the effect.

We proceed to apply the ad hoc method of sequentially solving the distribution and assignment problems. Each time the assignment problem is solved the $\{c_{pq}\}$ used in the distribution problems are updated. The following results are obtained:

Iteration	Access times $\{c_{pq}\}$				Distribution vector $\{f_{pq}\}$			
	c_{13}	c_{14}	c_{23}	c_{24}	f_{13}	f_{14}	f_{23}	f_{24}
1	1	1	1	1	75	25	75	25
2	394.3	125.2	387.2	120.0	59	41	91	9
3	309.4	205.4	470.4	43.2	100	0	50	50
4	525.3	1	258.6	238.3	50	50	100	0
5	263.4	249.0	515.5	1	100	0	50	50

Table 1

As it can easily be seen, the same distribution recurs at iteration 3 and iteration 5, which illustrates the lack of convergence of the ad hoc method.

The results of our highly distorted problem should not be extrapolated directly to draw conclusions about oscillations that may occur in real problems. Evidence suggests that γ should be close to the weighted average access times, which for our little example would be of the order of hundreds, rather than 1. Nevertheless the essential conclusion remains, and that is that repeating the ad hoc method several times will not result in better solutions to the combined problem.

4. Solution by application of the Frank-Wolfe algorithm

As we have mentioned previously, we subscribe to the approach take by Tomlin [14] and consider the use of Dantzig-Wolfe type decomposition as one good way of approaching the combined distribution assignment problem. We would like nevertheless to develop a different approach that also results effectively in a decomposition of the problem, without requiring the use of the "master problem" of linear programming computations.

We recall for the interested reader, that in [14] the traffic assignment model proposed calls for explicitly capacitated link flows such as

$$v_a = \sum_p \sum_q \sum_k \delta_{ak,pq} \, h_{k,pq} \leq b_a$$

where b_a is the capacity of arc a.

We shall not require such constraints, since the capacity of a link if implied by the volume-delay curve of a link. We proceed to adapt the Frank-Wolfe [7] algorithm to the combined problem (1) - (6). For simplicity, consider a prototype of our problem, that of minimizing a convex function subject to linear constraints:

$$\min \; Z(X) \quad \text{subject to} \quad AX = b \; ; \quad X \geq 0 \; . \tag{12}$$

The detailed steps of the Frank-Wolfe algorithm are as follows:

Step 1 Given a feasible solution X^1 , set $\ell = 1$.

Step 2 Determine Y^ℓ that minimizes

$$\nabla Z(X^\ell)\, Y \quad \text{subject to} \quad AY = b \; ; \quad Y \geq 0 \; .$$

Step 3 Set the descent direction $d^\ell = Y^\ell - X^\ell$

If $|\nabla Z(X^\ell)\, d^\ell| \leq \varepsilon$, terminate, where ε is a suitable convergence parameter.

Step 4 Find the optimal step length λ^ℓ that minimizes

$$Z(X^\ell + \lambda^\ell d^\ell) \quad \text{for} \quad 0 \leq \lambda \leq 1 \; .$$

Step 5 Revise the current solution

$$X^{\ell+1} = X^\ell + \lambda^\ell d^\ell \; ,$$

Set $\ell = \ell + 1$ and return to Step 2.

Consider now, the adaptation of the algorithm to the combined problem at an intermediate stage of the computations, when the solution is not yet optimal. A particular solution of the distribution problem, say $\{\bar{f}_{pq}\}$ and the path flows for an assignment, say $\{\bar{h}_{k,pq}\}$, are known. Then, in order to determine a descent direction one has to solve the problem

$$\text{Min} \sum_p \sum_q \left\{ \gamma \left(1 + \ell n \, \bar{f}_{pq}\right) y_{pq} + \sum_k \bar{u}_{k,pq} \, z_{k,pq} \right\} \quad (13)$$

Subject to

$$\sum_p y_{pq} = D_q \quad , \quad \text{all } q \quad (14)$$

$$\sum_q y_{pq} = O_q \quad , \quad \text{all } p \quad (15)$$

$$-y_{pq} + \sum_k z_{k,pq} = 0 \quad , \quad \text{all } (p,q) \quad (16)$$

$$y_{pq} \, , \, z_{k,pq} \geq 0 \quad (17)$$

The coefficients of the linear objective function (13) are the components of the gradient of the original objective function (1) with respect to $f_{p,q}$ and $h_{k,pq}$, evaluated at $\bar{f}_{pq}$, $\bar{h}_{k,pq}$, where $\bar{u}_{k,pq}$ is the length of the path k for O-D pair (p,q) when the flows are $\bar{h}_{k,pq}$. We require implicitly that $\bar{f}_{pq} > 0$ for all (p,q) in order to estimate the cost component $\ell n \, f_{pq}$, that would otherwise be undefined at $\bar{f}_{pq} = 0$ for some (p,q). Suppose that the calculations are initiated with $\bar{f}_{pq} > 0$, all (p,q); in any subsequent iteration, the distribution vector remains strictly positive if λ^{ℓ}, computed in Step 4 of the Frank and Wolfe algorithm, is strictly less than 1. In practice, λ^{ℓ} is unlikely to be near 1 if a reasonable solution serves to initiate the calculations. Another way to avoid this numerical problem is to modify slightly the objective function (1) to the form

$$\text{Min} \sum_p \sum_q \gamma \, f_{pq} \, \ell n \, (f_{pq}+1) + \sum_a S_a \, (v_a)$$

to ensure that the gradient is always defined at $\bar{f}_{pq} = 0$ for some (p,q).

Such a change implies a minor translation of the objective function that would not materially affect the solution values.

We proceed with the development of the algorithm. The linear program (13) - (17) simplifies[1] by noting that $\bar{u}_{k,pq}$ may be replaced by the length of the shortest path u^*_{pq} for each (p,q). In an extremal solution, all paths such that $\bar{u}_{k,pq} > u^*_{pq}$ would carry no flow, that is the corresponding $\bar{Z}_{k,pq} = 0$, and only one path flow will be positive and is one of the shortest paths of length u^*_{pq}. The objective (13) may be consequently written as

$$\text{Min} \sum_p \sum_q \{\gamma(1+\ell n\, f_{pq})\ y_{pq} + u^*_{pq} \sum_k Z_{k,pq}\} \qquad (18)$$

Now, by using equation (16) the problem of finding the descent direction becomes

$$\text{Min} \sum_p \sum_q \{\gamma(1+\ell n\, \bar{f}_{pq}) + u^*_{pq}\}\ y_{pq} \qquad (19)$$

$$\text{Subject to} \quad \sum_p y_{pq} = D_q \ , \ \text{all}\ q \quad \sum_q y_{pq} = 0_p \ , \ \text{all}\ p \qquad (20)$$

$$y_{pq} \geq 0$$

This is a linear program in the form of the classical Hitchcock problem. Its solution may be obtained very efficiently by one of the many algorithms available to solve this problem, such as [9].

We obtain next the $Z_{k,pq}$ by assigning the corresponding $\bar{y}_{pq}$ on the shortest route that was found while computing the u^*_{pq}. The descent direction will be defined as

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[1] This is the property which enables the efficient adaptation of the Frank-Wolfe algorithm to our problem.

$$\{\bar{y}_{pq} - \bar{f}_{pq} \;;\; \bar{z}_{k,pq} - \bar{h}_{k,pq}\} \quad \text{for all} \quad (p,q) \text{ and } (k,pq)$$

where $\bar{y}_{pq}$, $\bar{z}_{k,pq}$ are the solution to (13) - (17). In order to find a new improved solution, one has now to solve the one dimensional optimization problem

$$\text{Min} \sum_p \sum_q \{\gamma(1-\tau)\bar{f}_{pq} \ln [(1-\tau)\bar{f}_{pq}] + \gamma \tau y_{pq} \ln (\tau y_{pq})\}$$

$$+ \sum_a S_a\{(1-\tau)v_a+\tau w_a\}, \; 0\le\tau\le 1 \qquad (21)$$

$$\text{where} \quad v_a = \sum_p \sum_q \delta_{ak,pq} \bar{h}_{k,pq} \qquad (22)$$

$$w_a = \sum_p \sum_q \delta^*_{a,pq} \bar{y}_{pq} \qquad (23)$$

$$\text{and} \quad \delta^*_{a,pq} = \begin{cases} 1 & \text{if arc } a \text{ belongs to the shortest route for OD pair } (p,q) \\ 0 & \text{otherwise} \end{cases}$$

This problem may be easily solved by the Golden Section Method [13]. We summarize next the resulting algorithm.

Step 1 Select an initial distribution $\{f^o_{pq}\}$ and an initial assignment $\{v^o_a\}$; set $\ell = 0$

Step 2 Determine a descent direction by solving the transportation problem (19) - (20).

Step 3 If $\left| \sum_p \sum_q \{\gamma (1+\ln f^{\ell}_{pq}) + u^{*\ell}_{pq}\} (f^{\ell}_{pq} - y^{\ell}_{pq}) \right| \le \varepsilon$ for a suitable chosen convergence parameter ε , terminate, the solution is optimal.

Step 4 Solve the one-dimensional minimization problem (21) - (22) and compute the improved solutions

$$f_{pq}^{\ell+1} = f_{pq}^{\ell}(1-\tau^{\ell}) + \tau^{\ell} y_{pq}^{\ell}$$

$$v_a^{\ell+1} = v_a^{\ell}(1-\tau^{\ell}) + \tau^{\ell} w_a^{\ell} .$$

Set $\ell = \ell + 1$ and return to Step 2.

The application of the above algorithm to the problem of Figure 1 resulted in the distribution vector of {74, 26, 76, 24}, where we have rounded up the values to the nearest half integer, requiring 7 iterations. Since the assignment costs are unrealistically high compared to the distribution costs, we have reason to suspect that the convergence is slower than what may be normally expected. In practice, only a relatively small number of iterations may be required to obtain a consistent solution to the combined problem. The more congested the network, the more effect that the link costs will have on the distribution of traffic. The interested reader may find the solution in Figure 2.

5. Other Methods Proposed for the Combined Problem

In a recently completed doctoral dissertation, Evans [5] proposes a method for solving the combined problem, which consists in solving alternatively a distribution problem and finding shortest routes.

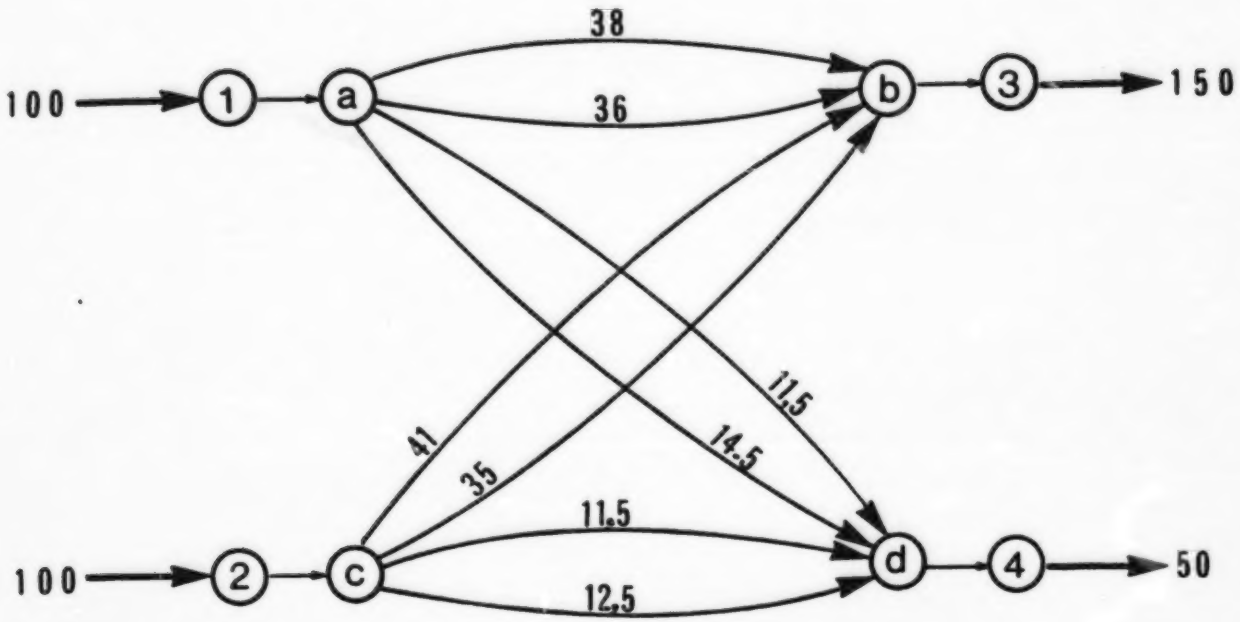

Figure 2

The algorithm is initiated with a particular solution of the combined problem; shortest routes are computed for each O-D pair and a new distribution problem is solved, using the length of the shortest routes as interzonal access times. The distribution problems are doubly constrained gravity models with exponential cost functions, that may be solved by the Furness [11] iterative procedure. The resulting distribution is assigned on the shortest paths found and then, a convex linear combination of the old and new distributions and assignments is found, which produces a new solution to the combined problem with a lower value of the objective function. The steps are repeated iteratively until suitable convergence is obtained. A rigorous proof of convergence is given. The author justifies the algorithm as an extension of previous work of Murchland [12] on the traffic assignment problem with elastic demand. The work of Evans resembles somewhat the algorithms developed by Irwin and Von Cube [10] for a transportation planning study of Toronto, Canada. Their work also allows for feedback between congested assignment and trip distribution, although they apply a sequential procedure. Starting from an initial solution of the distribution problem, the interzonal trips are assigned to the initial shortest routes. For successive iterations, new shortest routes are computed, and their lengths are used as access times for input to the distribution model. The new interzonal flows are then assigned in some proportion to the routes already found. The procedure is stopped when the interzonal travel times for successive iterations are quasi equal.

The method also incorporates proportional modal-split between road and transit. In essence, this algorithm resorts to a sequence of shortest route and trip distribution calculations, although the way in which the interzonal flows are assigned to the routes found is heuristic. Nevertheless these programs were employed in a large scale study and convergence to some approximate equilibrium values was claimed.

We would also like to point out some slight parellelism between the calculations required in the method proposed by Evans [5] and the decomposition algorithm of Tomlin [14], which also uses shortest route calculations and the Furness method for the trip distribution problem to generate columns for the "master problem".

In an unpublished paper, Bruynooghe [1], presents an algorithm for the combined distribution-assignment problem which is similar to the algorithm developed in Section 4. Bruynooghe's formulation of the combined problem is somewhat more general than the formulation that we present here. He obtains the formulation by imposing the usual trip production and attraction constraints (2), (3) to a formulation of a traffic assignment problem with elastic demand and flow dependent travel times. The method that he proposes may be interpreted as an adaptation of an algorithm of Bruynooghe, Gibert and Sakarovitch [2] for the assignment problem with elastic demand. The algorithm is stated directly and required a detailed proof of convergence.

In contrast to Bruynooghe's work, we obtain the solution method by the nontrivial adaptation of the Frank and Wolfe algorithm, which is a general mathematical programming method whose convergence properties are well known. Our approach permits the interpretation of the method within the framework of feasible direction algorithms of nonlinear programming.

6. Conclusion

In the preceding sections we have shown that sequential solution of the distribution and assignment problems may lead to invalid results. Several methods are available for solving the distribution and assignment of traffic simultaneously. The simplest of these methods typically require calculation of shortest routes, solution of either a sequence of Hitchcock problems or distribution problems and the minimization of a one dimensional convex function.

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